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POST BUCKLING ANALYSIS OF IMPERFECT NONLINEAR VISCOELASTIC COLUMNS

DAVID TOUATI and GABRIEL CEDERBAUM
The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel

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Abstract—The post buckling behavior of imperfect columns made of a nonlinear viscoelastic material is investigated. The material is modeled according to the Leaderman representation of nonlinear viscoelasticity. Solutions are developed to calculate the growth of the initial imperfection in time and within the elastica. The numerical results show that unlike the case of the post-buckling analysis of linear elastic or viscoelastic materials, here the ratio of h/l plays an important part in the structure response. It is shown that the post-buckling behavior of columns made of nonlinear viscoelastic materials is qualitatively and quantitatively better than in the linear viscoelastic case. Conclusions concerning creep buckling within the small deflection theory are also presented. © 1997 Elsevier Science Ltd.

INTRODUCTION

Stability of structures made of viscoelastic materials has a great importance in engineering design, especially in aerospace applications where the performance of structures made of polymer-based composites over a prolonged period of time has to be examined.

The buckling of linear viscoelastic columns has been extensively studied in the past. For example, Hilton (1952) has presented an analysis of a simply supported column with both arbitrary initial imperfections and arbitrary viscoelastic model. Kempner (1962) used a generalized form of the Maxwell model to characterize the buckling of nonlinear viscoelastic columns with small deflections. The effect of finite deformations on the stability of linear viscoelastic structures of simple rigid-bar-spring dashpot models was analyzed by Szyszkowski and Glockner (1985). Szyszkowski and Glockner (1986) evaluated the safe load for columns made of linear viscoelastic solid-type materials and geometrically nonlinear structures by various approximations. Comparison of geometrically linear and nonlinear creep buckling behaviors of eccentrically loaded linear viscoelastic columns was presented by Vinogradov (1985), where solutions were obtained by assuming that the deformation of the column is governed by the classical Bernoulli-Euler theory, and by means of the Schapery (1965) quasi-elastic methods. The Vinogradov investigation of the nonlinear geometrical problem predicts that (a) no infinite increase of lateral deflections with time (contrary to the linear theory), and (b) a satisfactory agreement between the results of the linear and the nonlinear geometrical analysis is achieved only in the case of relatively small deformations. Vinogradov (1985), and later on in (1987), showed also that in the case of limited creep there is a safe load limit below which the creep buckling process stabilizes in time. Creep buckling of nonlinear viscoelastic columns was analyzed by Chung and Chung-Li (1986), using a nonlinear single integral for modeling nonlinear generalized Kelvin materials. The stability of an initially imperfect nonlinear viscoelastic column was investigated by Distefano and Sackman (1968) by using the constitutive relation represented by a Volterra-Frechet functional polynomial.

From the above investigations it follows that beyond a certain safe load level, large deflections are developed, and can generate structural changes in the material, so that other nonlinear effects should be considered (see Vinogradov (1985)).

In this paper, we investigate the behavior of columns which are geometrically nonlinear (elastica), made of *nonlinear* viscoelastic material, and have initial imperfection. The material constitutive equation is given by a single integral, according to the Leaderman

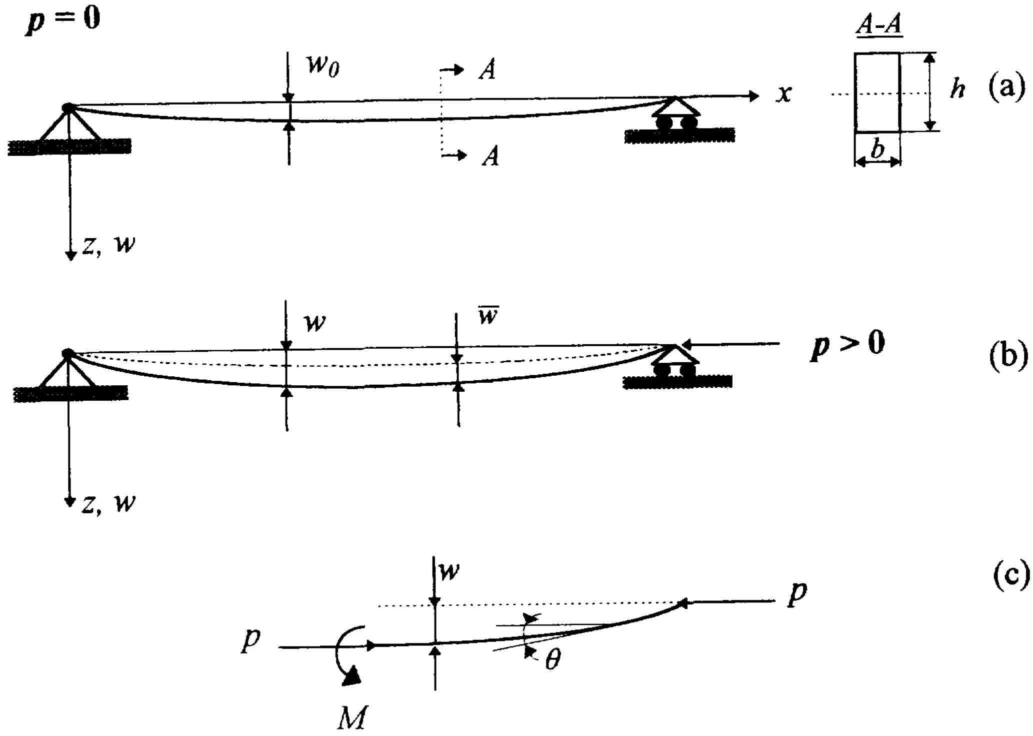

Fig. 1. Buckling of imperfect columns.

(1962) model for nonlinear viscoelastic materials. This model was compared e.g., by Smart and Williams (1972) using the Schapery (1969) and the BKZ (1965) models, and one of their main conclusions was that the Leaderman model is a very useful representation where prediction and simplicity are concerned.

PROBLEM FORMULATION

An initially imperfect, simply supported column of length l and cross sectional area A is subjected to a constant compressive load, p, as shown in Fig. 1.

The mechanical properties of the material are defined by the single integral representation of nonlinear viscoelastic theory, as given by Leaderman (1962). In this model, the stress-strain constitutive relation is given by

$$\sigma(t) = E(0)g[\varepsilon(t)] + \int_{0^-}^{t} \dot{E}(t-\tau)g[\varepsilon(\tau)]\,d\tau \tag{1}$$

where $g(\varepsilon)$ is given by

$$g(\varepsilon) = \varepsilon + \beta\varepsilon^2 + \gamma\varepsilon^3 \ldots \tag{2}$$

which for small strains, $g(\varepsilon) \to \varepsilon$. $E(t)$ is a time-dependent relaxation function, given by the Perony series

$$E(t) = E_\infty + \sum_{i=1}^{n} E_i e^{-\alpha_i t} \tag{3}$$

which at $t = 0$ denotes the initial Young's modulus of the material E_0. E_∞ is the value of the relaxation function at infinite time and E_i and α_i are constants ($1/\alpha_i$ represents the relaxation time).

The equilibrium equation of the deflected column (see Fig. 1c) is

$$M = -pw \tag{4}$$

where the bending moment is

$$M(t) = -b\int_{-h/2}^{h/2} z\sigma(t)\,\mathrm{d}z \tag{5}$$

and $\alpha(t)$ is given by eqn (1).

The bending strain, ε, in a fiber at distance z from the neutral axis is given by

$$\varepsilon = -z\theta_{,x} \tag{6}$$

where θ is the angle of rotation of the column cross-section. Within the elastica

$$\theta = \sin^{-1}(\bar{w}_{,x}) \tag{7}$$

where $\bar{w} = w - w_0$ and w_0 is a small imperfection (see Fig. 1).

By substituting eqns (2) and (6) into (1), and using the resulting relation for rewriting the bending moment (eqn (5)) one obtains

$$M(t) = E_0[I\theta_{,x} + \gamma I_1\theta_{,x}^3] + \int_{0^-}^{t} \dot{E}(t-\tau)[I\theta_{,x} + \gamma I_1\theta_{,x}^3]\,\mathrm{d}\tau \tag{8}$$

where $I = bh^3/12$ and $I_1 = bh^5/80$.

From eqn (7) one obtains

$$\theta_{,x} = \frac{\bar{w}_{,xx}}{\cos\theta} \tag{9}$$

where $\cos\theta = \sqrt{1-\sin^2\theta} \cong 1 - \frac{1}{2}\bar{w}_{,x}^2$, so that

$$\frac{1}{\cos\theta} \cong 1 + \frac{1}{2}\bar{w}_{,x}^2 \tag{10}$$

and the expressions derived for $\theta_{,x}$ and $\theta_{,x}^3$ are thus given by

$$\theta_{,x} \cong \bar{w}_{,xx} + \tfrac{1}{2}\bar{w}_{,x}^2\,\bar{w}_{,xx} \tag{11}$$

$$\theta_{,x}^3 \cong \bar{w}_{,xx}^3(1 + \tfrac{3}{2}\bar{w}_{,x}^2). \tag{12}$$

Substituting eqns (11–12) into eqn (8) yields the following form of eqn (4)

$$\bar{w}_{,xx} + \frac{1}{2}\bar{w}_{,xx}\bar{w}_{,x}^2 + \frac{\gamma I_1}{I}\bar{w}_{,xx}^3 + \frac{p}{E_0 I}w = -\int_{0^-}^{t} \dot{D}(t-\tau)\left(\bar{w}_{,xx} + \frac{1}{2}\bar{w}_{,xx}\bar{w}_{,x}^2 + \frac{\gamma I_1}{I}\bar{w}_{,xx}^3\right)\mathrm{d}\tau \tag{13}$$

where

$$\dot{D}(t) = \frac{\dot{E}(t)}{E_0},$$

and high order terms were omitted.

We introduce now the following dimensionless quantities into the above equation

$$\xi = \frac{x}{l}, 0 \leqslant \xi \leqslant 1$$

$$\bar{\omega} = \frac{\bar{w}}{l}, \quad \omega = \frac{w}{l}, \quad \omega_0 = \frac{w_0}{l}, \quad \eta = \frac{h}{l} \tag{14}$$

to obtain

$$\bar{\omega}_{,xx} + \frac{1}{2}\bar{\omega}_{,xx}\bar{\omega}_{,x}^2 + \frac{3\gamma}{20}\eta^2\bar{\omega}_{,xx}^3 + \frac{p}{p_e}\pi^2\omega = -\int_{0^-}^{t} \dot{D}(t-\tau)\left(\bar{\omega}_{,xx} + \frac{1}{2}\bar{\omega}_{,xx}\bar{\omega}_{,x}^2 + \frac{3\gamma}{20}\eta^2\bar{\omega}_{,xx}^3\right)\mathrm{d}\tau \tag{15}$$

where

$$p_e = \frac{E_0 I\pi^2}{l^2}\left(= \frac{E_0 A\pi^2}{12}\eta^2\right)$$

is the Euler buckling load.

Equation (15) represents a nonlinear initial and boundary valve problem, which can be solved within the assumption that the time response of the first mode dominates the total column response (see, e.g., Minahen and Knauss (1993)). Thus the approximate solution for eqn (15) can be obtained by considering the following function

$$\omega = f(t)\sin\pi\xi \tag{16}$$

and a sinusoidal initial imperfection function

$$\omega_0 = e\sin\pi\xi \tag{17}$$

where e is a small positive constant and $f(t)$ is an unknown time dependent function of the central span deflection, to be found. Substituting eqns (16–17) into (15), notice that

$$\bar{\omega} = (f(t) - e)\sin\pi\xi \tag{18}$$

and since only the first mode is considered

$$\sin^3\pi\xi \cong \tfrac{3}{4}\sin\pi\xi. \tag{19}$$

Equation (15) is now written in the following form

$$\left(1 - \frac{p}{p_e}\right)f(t) - e + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)(f(t) - e)^3 =$$

$$-\int_{0^-}^{t} \dot{D}(t-\tau)\left[(f(\tau) - e) + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)(f(\tau) - e)^3\right]\mathrm{d}\tau. \tag{20}$$

Equation (20) is a nonlinear integral equation which governs the finite deflection response of the nonlinear viscoelastic column subjected to axial loading.

METHOD OF SOLUTION

Consider a material for which the relaxation function is given by eqn (3) and $n = 1$

$$E(t) = E_{\infty} + E_1 e^{-\alpha t} \tag{21}$$

where E_1, E_{∞} and α are constants, to be given for the material considered. Then

$$D(t) = \frac{E(t)}{E_0} = \frac{E_{\infty} + E_1 e^{-\alpha t}}{E_{\infty} + E_1} \quad \text{and} \quad \dot{D}(t) = -\frac{\alpha E_1 e^{-\alpha t}}{E_{\infty} + E_1}. \tag{22}$$

Substituting eqn (22) into eqn (20) and then differentiating via the Leibnitz's rule, the following ordinary differential equation is derived

$$\frac{df}{dt} = \alpha \frac{\left(\frac{p}{p_e} - \frac{E_{\infty}}{E_0}\right) f + \frac{E_{\infty}}{E_0} e - \frac{E_{\infty}}{E_0}\left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)(f-e)^3}{1 - \frac{p}{p_e} + \left(\frac{3\pi^2}{8} + \frac{27\gamma\pi^4}{80}\eta^2\right)(f-e)^2} \tag{23}$$

where $E_0 = E_{\infty} + E_1$. The initial condition for eqn (23) is the instanteneous static deflection f_0, at $t = 0^+$ and is given by the relation

$$\frac{p}{p_e} = 1 - \frac{e}{f_0} + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)\frac{(f_0 - e)^3}{f_0}. \tag{24}$$

Equation (23) can be solved numerically, or analytically for special cases.

We note here that relation (24) of the instantaneous static deflection represents the *elastica* type post-buckling of columns made of nonlinear elastic 'hard materials' with initial imperfection. In this case the constitutive equation is of the form $\sigma = E(\varepsilon + \kappa\varepsilon^3)$.

NUMERICAL RESULTS AND DISCUSSION

First it is recognized that the buckling equation of *linear* viscoelastic column based on the *small deflection* theory, as derived from (23), is of the form

$$\frac{df}{dt} = \alpha \frac{\left(\frac{p}{p_e} - \frac{E_{\infty}}{E_0}\right) f + \frac{E_{\infty}}{E_0} e}{1 - \frac{p}{p_e}} \tag{25}$$

and the solution of this equation is

$$f = \frac{e}{1 - p/p_{\infty}}\left\{1 + \frac{p/p_e - p/p_{\infty}}{1 - p/p_e} \exp\left[-\alpha\frac{p_{\infty}}{p_e}\left(\frac{1 - p/p_{\infty}}{1 - p/p_e}\right)\right] t\right\} \tag{26}$$

where $p_{\infty} = E_{\infty} I\pi^2/l^2$ $(=(E_{\infty} A\pi^2/12)\eta^2)$. We note that the initial value in this case is the loading-deflection relationship of *elastic* column with initial imperfection, $p/p_e = 1 - (e/f_0)$.

For $p < p_{\infty}$ (p_{∞} is defined as a safe load limit), the response of the column is limited to an asymptotic value f_{∞}, as $t \to \infty$, obtained at $df/df \to 0$, and is given by the relation

$$f_\infty = \frac{e}{1-\dfrac{p}{p_\infty}} \tag{27}$$

and the column is considered to be stable. When $p_\infty < p < p_e$ the response will increase exponentially (see eqn (26)), and, finally, creep instability will occur. The instantaneous buckling occurs when $p \geqslant p_e$ and the column loses stability instantly, at $t = 0$ (see also Vinogradov (1987), Bazant and Cedolin (1991) and Minahen and Knauss (1993)).

In the *nonlinear* case, when the initial imperfection is zero, one obtains from eqn (23)

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \alpha \frac{\left(\dfrac{p}{p_e} - \dfrac{E_\infty}{E_0}\right) f - \dfrac{E_\infty}{E_0}\left(\dfrac{\pi^2}{8} + \dfrac{9\gamma\pi^4}{80}\eta^2\right) f^3}{1 - \dfrac{p}{p_e} + \left(\dfrac{3\pi^2}{8} + \dfrac{27\gamma\pi^4}{80}\eta^2\right) f^2} \tag{28}$$

which for $p > p_e$ has the following solution

$$2\alpha \frac{p_\infty}{p_e} t = 3 \ln \frac{f_0^2 + A_1}{f^2 + A_1} + A_2 \ln \frac{f_0^2 (f^2 + A_1)}{f^2 (f_0^2 + A_1)} \tag{29}$$

where

$$A_1 = \frac{1 - \dfrac{p}{p_\infty}}{\dfrac{\pi^2}{8} + \dfrac{9\gamma\pi^4}{80}\eta^2}, \quad A_2 = \frac{1 - \dfrac{p}{p_e}}{1 - \dfrac{p}{p_\infty}}, \quad f_0^2 = \frac{\dfrac{p}{p_e} - 1}{\dfrac{\pi^2}{8} + \dfrac{9\gamma\pi^4}{80}\eta^2} \tag{30}$$

and for $p = p_e$

$$f^2 = \frac{1 - \dfrac{p_e}{p_\infty}}{\dfrac{\pi^2}{8} + \dfrac{9\gamma\pi^4}{80}\eta^2}\left[\exp\left(-\frac{2p_\infty}{3p_e} t\right) - 1\right]. \tag{31}$$

The asymptotic value of the deflection in the case of (28) is given by

$$f_\infty = \sqrt{\frac{\dfrac{p}{p_\infty} - 1}{\dfrac{\pi^2}{8} + \dfrac{9\gamma\pi^4}{80}\eta^2}}. \tag{32}$$

In the following, we consider the most general form of eqn (23). The asymptotic value of the deflection here is

$$\frac{p}{p_e} = \frac{E_\infty}{E_0}\left[1 - \frac{e}{f_\infty} + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)\frac{(f_\infty - e)^3}{f_\infty}\right]. \tag{33}$$

We would like to note here that: (a) by increasing the value of the relaxation time, $1/\alpha_i$, the time needed to reach the asymptotic value of the deflection is decreased, and (b) unlike the case of the post-buckling analysis of linear elastic or viscoelastic materials, here $\eta (= h/l)$ plays a very important part in the structure behavior (see also Figs 3 and 6 below).

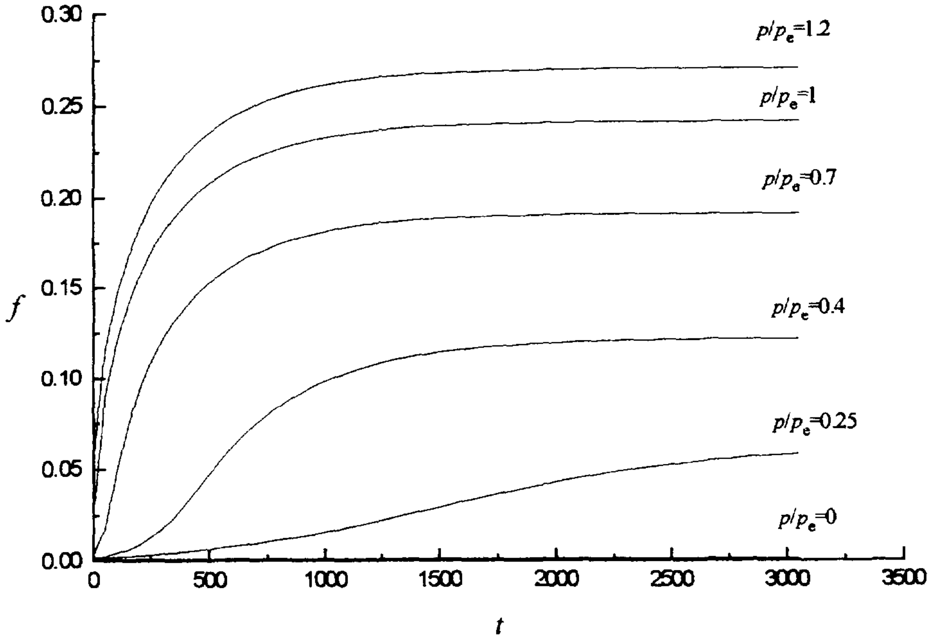

Fig. 2. The deflection $f(t)$ vs t for $\eta = 0.05$ and different ratios of p/p_e.

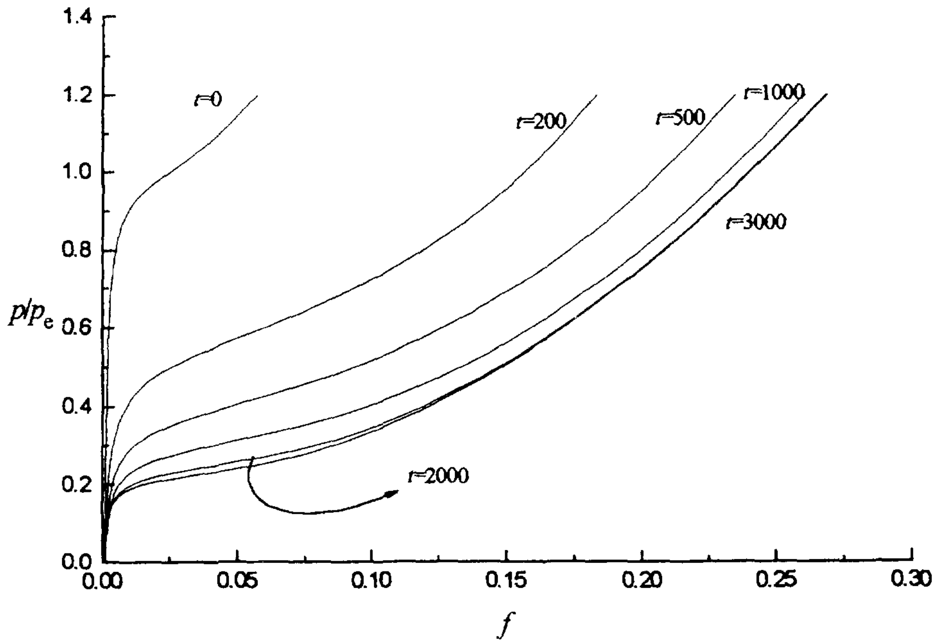

Fig. 3. p/p_e vs f for $\eta = 0.05$ at different values of t.

The numerical results were obtained by using $E_\infty = 0.2$, $E_1 = 0.8$, $\alpha = 0.02$, $e = 0.001$ and $\gamma = 2500$, and within the high order Runge–Kutta method.

Figure 2 shows the deflection $f(t)$ vs t for $\eta = 0.05$ at different ratios of p/p_e, and in Fig. 3 p/p_e vs f is given, for $\eta = 0.05$ and at different times. From these figures it is seen that the deflections gradually increase to their asymptotic value as given by eqn (33). This result indicates that contrary to the linear theory, the nonlinear analysis predicts no infinite increase of the lateral deflections with time. The same conclusion was indicated by Vinogradov (1985), concerning the creep buckling analysis of geometrically nonlinear viscoelastic columns.

The variation of $f(t)$ vs t for two cases of loadings is given in Fig. 4, as obtained by considering linear (derived by substituting $\gamma = 0$ in eqn (23)) and nonlinear viscoelastic

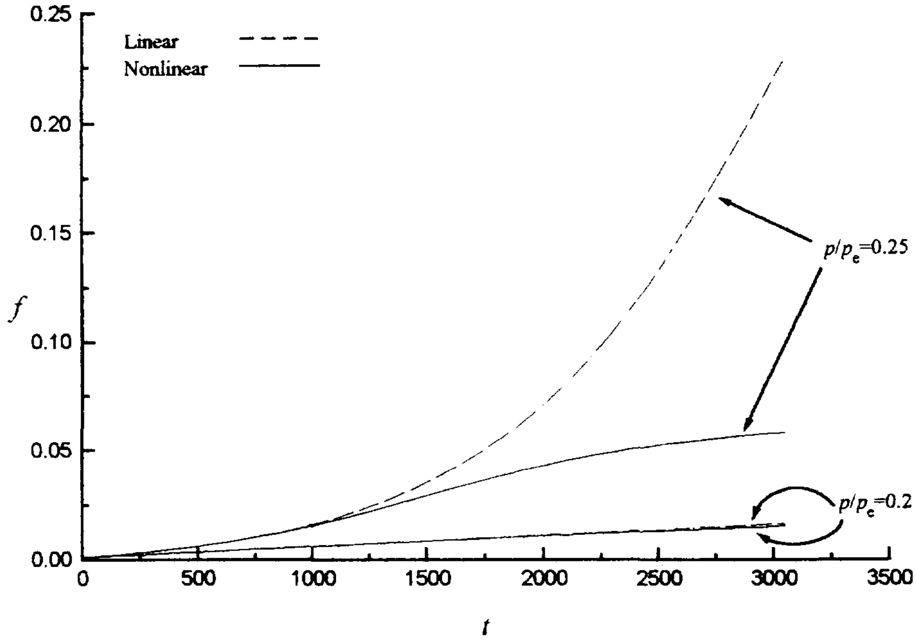

Fig. 4. The deflection $f(t)$ vs t for different ratios of p/p_e, in the linear and nonlinear viscoelastic cases, and $\eta = 0.05$.

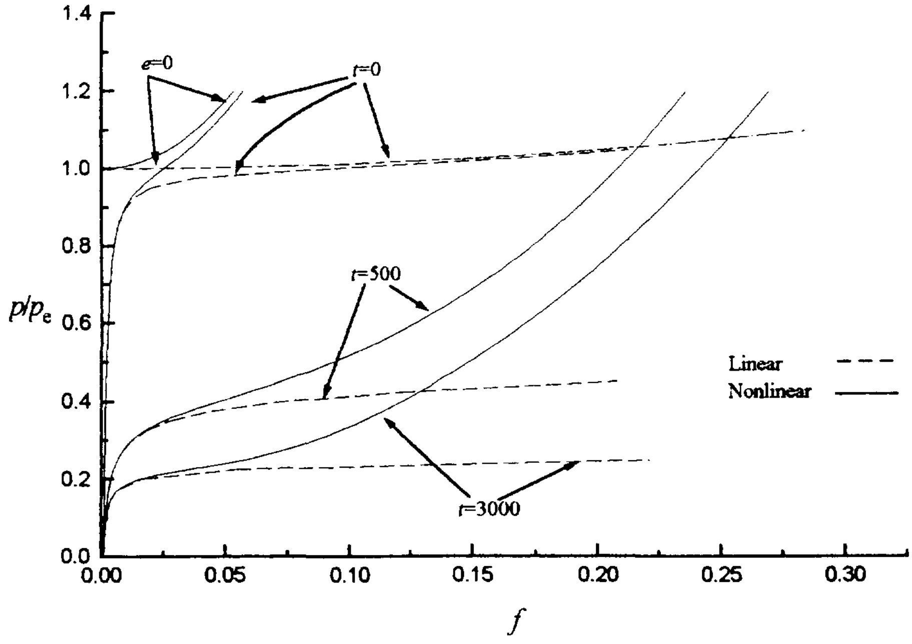

Fig. 5. p/p_e vs f for different values of t, in the linear and nonlinear viscoelastic cases, for $\eta = 0.05$.

materials. These cases are considered also in Fig. 5, where the variation of p/p_e vs f is shown. It can be seen that the differences in the results are increasing with the loading and with the time. Those differences are not only quantitative, but qualitative as well. For example, for $p/p_e = 0.2$ in Fig. 5, at $t = 3000$ (and also in Fig. 4 for $p/p_e = 0.25$), the response in the linear case is practically unstable, while in the nonlinear case it is bounded. This can be seen also in the instantaneous response (at $t = 0$), for both perfect ($e = 0$) and imperfect columns. By "practically unstable" in the linear viscoelastic case we mean that the response is sharply increased with any small change in the loading (as can be seen, for example, for the case of $p/p = 0.25$ in Fig. 4 at high values of t, and in Fig. 5 at $t = 3000$). These results indicate that the post-buckling response of columns made of nonlinear viscoelastic materials exhibit better behavior than of the linear viscoelastic counterparts. The improvement is

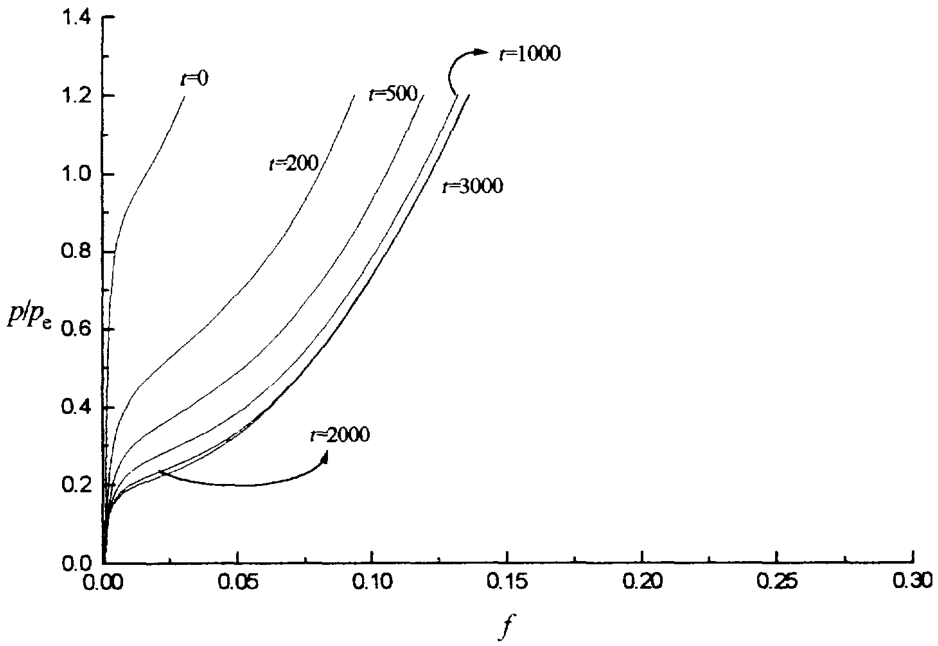

Fig. 6. p/p_e vs f for $\eta = 0.1$ at different values of t.

seen even at relatively short times, and for relatively small (and thus more interesting) values of the response. The same, of course, can be said for a perfect columns.

Next, we are interested in investigating the importance of $\eta(= h/l)$ in the post buckling behavior of the column. As can be seen from eqn (23), the linear viscoelastic case (obtained by setting $\gamma = 0$) is independent of η, and that η is affecting only the material nonlinear component and not of the geometrical one. We examine this phenomenon in Fig. 6, where we considered the same case as in Fig. 3 but for a thicker column, $\eta = 0.1$. Here, the increasing of η adds more nonlinearity to the system and thus the deflection decrease, as can be seen by comparing Fig. 3 with Fig. 6. Thus, we conclude that the effect of the nonlinear viscoelasticity is even more pronounced for thicker structures.

Finally, we would like to note that the same problem can be considered by keeping f as a constant in eqn (20). This yields an expression for the time dependent loading, which in our case is

$$\frac{p(t)}{p_e} = \left[1 - \frac{e}{f} + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)\frac{(f-e)^3}{f}\right]D(t). \tag{34}$$

In the elastic case ($\alpha = 0$), we obtain the elastica type critical loading for imperfect columns made of nonlinear (hard) material

$$p = p_e\left[1 - \frac{e}{f} + \left(\frac{\pi^2}{8} + \frac{9\gamma\pi^4}{80}\eta^2\right)\frac{(f-e)^3}{f}\right]. \tag{35}$$

Other sub-cases can also be obtained from eqn (34).

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